Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

December 2013 Mat 3172 Professor M. Alvo

Time: 3 hours

Student name: Student number:

This is an open book exam. Calculators are permitted. Answer all the questions. The exam is out of 80 points.

1. [10] Suppose that the life distribution of an item has the hazard rate $\lambda(t) = t^2, t > 0$. What is the probability that

a) the item survives to age 2?

b) the item's lifetime is between 0.4 and 1.4?

c) a 1-year old item will survive to age 2?

2. [10] X and Y are two independent random variables having geometric distributions with parameter p. Recall

$$P(X = x) = p(1-p)^{x-1}, x = 1, 2, .$$

a) Find the density of $Z = min\ (X, Y)$

b) Find the density of the sum $(X + Y)$.

c) Calculate $P(Y \geq X)$

d) Calculate EZ

3. [6] A coin has probability p of turning up heads. It is flipped until either heads or tails has occurred three times. What is the expected number of flips?

(Hint: Recall the negative binomial distribution)

4. [10] Let $X_1, ..., X_n$ be a random sample from a uniform distribution on the interval $(0, \theta)$.

a) If n is an odd integer, find the density of the median of the sample.

b) Show that the median of the sample converges in probability to the median of the population as $n \to \infty$.

Hint: The Beta density is given by

$$\begin{aligned} f(x) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1, \alpha > 0, \beta > 0 \\ mean &= \frac{\alpha}{\alpha+\beta},\ variance = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \end{aligned}$$

5. [12] If

$$\begin{aligned} f(x,y) &= c(y-x)^{\alpha},\ 0 \le x < y \le 1 \\ &= 0, otherwise \end{aligned}$$

a) find the value of the constants c, α which make f a proper joint density of X and Y.

b) Find the marginal densities of X and Y.

c) What is the conditional density of X given Y?

d) What is the conditional expectation of X given Y?

6. [10] Let X be a random variable having a log normal distribution; that is, $Y = \ln X$ follows a $N(\mu, \sigma^2)$.

2 a) Use the Moment generating function to show that $EX = exp\left(\mu + \frac{\sigma^2}{2}\right)$.

4 *b*) If Z has continuous density

$$f(z) = \frac{1}{2} \exp - |z|, -\infty < z < \infty$$

show that its moment generating function is given by $\frac{1}{1-t^2}, -1 < t < 1$.

4 *c*) Compute $EZ^{2n}, n = 1, 2, ...$

7.[10] Suppose that X, Y, are two independent random variables having exponential distributions with mean 1. Find

a) the joint density of $U = X + Y, V = \frac{X}{Y}$.

b) the marginal density of U

c) the marginal density of V

d) the expectation of V.

8. [12] Let X be a Poisson distributed random variable with mean λ.

a) Use Markov's inequality to obtain an upper bound for $p = P(X \geq c)$ as a function of λ, c.

b) Use Chebyshev's inequality to obtain an upper bound for p.

c) Use Chernoff's inequality to obtain an upper bound for p.

d) Approximate p by means of the Central limit theorem.

Final Solutions   MAT 3172   MAYER ALVO

1. $F(t) = 1 - \exp - \int_0^t \lambda(s)\,ds = 1 - \exp - \int_0^t s^2\,ds = 1 - e^{-t^3/3}$

a) $P(X > 2) = e^{-2^3/3} = e^{-8/3}$

b) $P(0.4 < X < 1.4) = F(1.4) - F(0.4) = e^{-(0.4)^3/3} - e^{-(1.4)^3/3}$

c) $P(X > 2 \mid X > 1) = \dfrac{P(X>2)}{P(X>1)} = \dfrac{e^{-8/3}}{e^{-1/3}} = e^{-7/3}$

2. a) $P(\min(X, Y) \geq z) = P(X \geq z)\,P(Y \geq z) = (1-p)^{2(z-1)} = \left[(1-p)^2\right]^{z-1}$

$\Longrightarrow$ Geometric(q) where $q = 1 - (1-p)^2$

b) $P(X+Y=z) \underset{\text{Convolution}}{=} \sum_{x=0}^{z} P(X=x, Y=z-x)$

$= \sum_{x=0}^{z} p(1-p)^{x-1} p(1-p)^{z-x-1} = p^2(1-p)^{z-2} \sum_{x=0}^{z} 1 = (z+1)\,p^2(1-p)^{z-2}$

$z = 2, 3, \ldots$

c) $P(Y \geq X) = \sum_{y \geq x} p^2 (1-p)^{x-1} (1-p)^{y-1}$

$= p^2 \sum_{x=1}^{\infty} \sum_{y=x}^{\infty} (1-p)^{x-1}(1-p)^{y-1} = p^2 \sum_{x=1}^{\infty} (1-p)^{x-1} \dfrac{(1-p)^{x-1}}{p}$

$= p \sum_{x=1}^{\infty} \left[(1-p)^2\right]^{x-1} = p \dfrac{1}{1-(1-p)^2} = \dfrac{1}{2-p}$

d) $EZ = \dfrac{1}{q}$

3. We use the negative binomial. Consider first having 3 heads.
Let X = # of tosses required for 3 heads.

$$P(X=3) = \binom{3-1}{3-1} p^3 (1-p)^0 = p^3 \qquad = 1/8$$

$$P(X=4) = \binom{4-1}{3-1} p^3 (1-p) = 3\, p^3 (1-p) = 3/16$$

$$P(X=5) = \binom{5-1}{3-1} p^3 (1-p)^2 = 6\, p^3 (1-p)^2 = 6/32$$

We don't go beyond 5 tosses since then we would get 3 tails.

By symmetry since the coin is fair, we get the same probabilities for tails. Hence, if Y = # of tosses required for 3 heads or 3 tails

$$P(Y=3) = 2(1/8), \quad P(Y=4) = 2(3/16), \quad P(Y=5) = 2(6/32).$$

$$EY = \sum_{3}^{5} y\, P(Y=y) = \frac{6}{8} + \frac{24}{16} + \frac{60}{32} = \frac{33}{8}$$

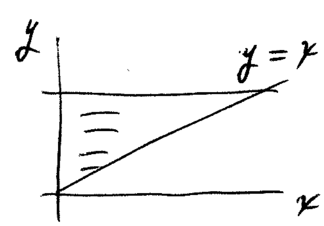
4. a) Set $n = 2k+1$ $\qquad f_X(x) = \dfrac{n!}{k!\,1!\,k!} f(x)\, F^k(x)\,(1-F(x))^k$

$$= \frac{n!}{k!\,k!} \left(\frac{x}{\theta}\right)^k \left(1 - \frac{x}{\theta}\right)^k$$

If $Y = \dfrac{X}{\theta}$, then $Y \sim \text{Beta}(k+1, k+1)$, $EY = \dfrac{1}{2}$, $V(Y) = \dfrac{1}{4(2k+2)}$

b) $$P\left(\left|Y - \tfrac{1}{2}\right| > \varepsilon\right) \le \frac{V(Y)}{\varepsilon^2} = \frac{1}{4\varepsilon^2 (n+1)} \longrightarrow 0 \text{ as } n \to \infty$$

5. a) $$1 = c\int_0^1 \int_0^y (y-x)^\alpha \, dx\, dy$$



$$= c\int_0^1 \left. \frac{(y-x)^{\alpha+1}}{\alpha+1} \right|_0^y dy$$

$$= \frac{c}{(\alpha+1)(\alpha+2)}$$

$$\Rightarrow \quad c = (\alpha+1)(\alpha+2)$$

$\Rightarrow \quad c = (\alpha+1)(\alpha+2) \qquad \alpha \neq -1, \neq -2, \alpha+1>0, \alpha+2>0$

and hence $\quad c > -1$

b) Marginal of Y $\qquad c\int_0^y (y-x)^\alpha dx = (\alpha+2)\, y^{\alpha+1} \qquad 0<y<1$

Marginal of X $\qquad c\int_x^1 (y-x)^\alpha dy = (\alpha+2)(1-x)^{\alpha+1} \qquad 0<x<1$

c) $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)} = (\alpha+1)\dfrac{(y-x)^\alpha}{y^{\alpha+1}} \qquad 0 \le x < y < 1$

d) $E[X|Y] = \int_0^y x\, f_{X|Y}(x|y)\,dx = (\alpha+1)\, y \int_0^1 u(1-u)^\alpha du$

$= (\alpha+1)\dfrac{y}{c} = \dfrac{y}{\alpha+2}$

Beta integral

6. a) $Y = \ln X \Rightarrow X = e^Y$

$EX = Ee^Y = M_Y(1) = e^{\mu+\sigma^2/2}$

b) $M_Y(t) = \frac{1}{2}\int_{-\infty}^{\infty} e^{ty} e^{-|y|} dy = \frac{1}{2}\left[\int_{-\infty}^0 e^{ty} e^{y} dy + \int_0^{\infty} e^{ty} e^{-y} dy\right]$

$= \frac{1}{2}\left[\frac{1}{1+t} + \frac{1}{1-t}\right] = \frac{1}{1-t^2} \qquad -1<t<1$

c) $M_Y'(t) = \frac{1}{2}\left[-\frac{1}{(1+t)^2} + \frac{1}{(1-t)^2}\right] = 0$

$M_Y^{2n}(t) = \frac{1}{2}\left[(2n)!\,(1+t)^{-2n-1} + (2n)!\,(1-t)^{-2n-1}\right]$

$= (2n)! = EY^{2n}$

7 a)

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = -\frac{x}{y^2} - \frac{1}{y} = -\frac{(x+y)}{y^2}$$

$$u = x+y, \quad v = x/y \implies x = \frac{uv}{v+1}, \quad y = \frac{u}{v+1}$$

$$f(x,y) = e^{-(x+y)}$$

$$f(u,v) = e^{-u} |J|^{-1} = e^{-u} \cdot \frac{u}{(v+1)^2}, \quad 0 < \frac{uv}{v+1} < \infty, \quad 0 < \frac{u}{v+1} < \infty$$

b) $f_U = u e^{-u}, u > 0 \quad \text{Gamma}(2,1)$

c) $f_V = \frac{1}{(v+1)^2}, \quad v > 0$

$\therefore$ U, V are indep.

d) $EV = \int_0^\infty \frac{v}{(v+1)^2} dv \approx \ln(v+1) \Big|_1^\infty = \infty$

8. a) $P(X \geq c) \leq \frac{EX}{c} = \frac{\lambda}{c}$ (see p 367)

b) $P(X \geq c) \leq \frac{\sigma^2}{\sigma^2 + c^2} = \frac{\lambda}{\lambda + c^2}$ (see p 382)

c) $P(X \geq c) \leq e^{-\lambda} \left(\frac{e\lambda}{c}\right)^c$ (see p 385)

d) Let $Y \sim \text{Poisson}(\lambda/n)$ $\quad X = \sum_1^n Y_i, \quad EY_i = \frac{\lambda}{n} = \mu, V(Y_i) = \frac{\lambda}{n} = \sigma^2$

$$P(X \geq c) = P\left(\frac{\sum_1^n Y_i - n\mu}{\sqrt{n\sigma^2}} \geq \frac{c + 0.5 - n\mu}{\sqrt{n\sigma^2}}\right)$$

$$\cong 1 - \Phi\left(\frac{c + 0.5 - \lambda}{\sqrt{\lambda}}\right)$$

Notes $P(\min(X,Y) = z) = P(X = z, Y > z) + P(X > z, Y = z) - P(X = z, Y = z)$

$$= 2\,p(1-p)^{z-1}\cdot(1-p)^{z-1} - p^2(1-p)^{2(z-1)}$$

$$= p\,(1-p)^{2(z-1)}\left[\,2 - p\,\right]$$